

# BINDING CONDITION FOR A GENERAL CLASS OF QUANTUM FIELD HAMILTONIANS

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**ABSTRACT.** We consider a system of a quantum particle interacting with a quantum field and an external potential  $V(\mathbf{x})$ . The Hamiltonian is defined by a quadratic form  $H^V = H^0 + V(\mathbf{x})$ , where  $H^0$  is a quadratic form which preserves the total momentum.  $H^0$  and  $H^V$  are assumed to be bounded from below. We give a criterion for the positivity of the binding energy  $E_{\text{bin}} = E^0 - E^V$ , where  $E^0$  and  $E^V$  are the ground state energies of  $H^0$  and  $H^V$ . As examples of the result, the positivity of the binding energy of the semi-relativistic Pauli-Fierz model and Nelson type Hamiltonian is proved.

## 1. INTRODUCTION

We consider a Hamiltonian of the form

$$(1) \quad H^V = H^0 + V \otimes I,$$

acting on the Hilbert space  $\mathcal{H} = L^2(\mathbb{R}^d; d\mathbf{x}) \otimes \mathcal{K}$ , where  $\mathcal{K}$  is a Hilbert space,  $H^0$  is a semi-bounded quadratic form on  $\mathcal{H}$  and  $V$  is the operator of multiplication by a real function  $V(\mathbf{x})$  in  $L^2(\mathbb{R}^d; d\mathbf{x})$ . We are interested in the ground state energy  $E^V$  of  $H^V$ . The *binding energy* of the system is defined by

$$(2) \quad E_{\text{bin}} = E^0 - E^V.$$

In this paper, we give a criterion for  $E_{\text{bin}}$  to be strictly positive.

Hamiltonians of the form (1) appear in models of a quantum particle interacting with a quantum field. One of the important examples is the *Pauli-Fierz Hamiltonian*, for which  $d = 3$ ,  $\mathcal{K}$  is the bosonic Fock space over  $L^2(\mathbb{R}^3 \times \{1, 2\})$  and

$$(3) \quad H^0 = H_{\text{PF}}^0 := \frac{1}{2m}(\mathbf{p} \otimes I + \sqrt{\alpha} \mathbf{A}(\mathbf{x}))^2 + I \otimes H_f$$

where  $H_f$  is the free photon energy,  $\alpha$  is the fine structure constant,  $\mathbf{A}(\mathbf{x})$  is the quantized vector potential and  $V(\mathbf{x})$  is the nuclear potential (see [3]). The positivity of the binding

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energy is used as a hypothesis to establish the existence of a ground state of the Pauli-Fierz model in [3]. In [3] the positivity of the binding energy is obtained by assuming that

$$(4) \quad \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})$$

has a negative energy ground state. In this paper, we generalize the method developed in [3] and apply it to several types of quantum field Hamiltonians such that the semi-relativistic Pauli-Fierz Hamiltonian, the Pauli-Fierz Hamiltonian with dipole approximation and Nelson type Hamiltonians.

## 2. DEFINITIONS AND MAIN RESULTS

If  $\mathcal{H}$  is a Hilbert space we denote by  $(\cdot|\cdot)_{\mathcal{H}}$  the scalar product on  $\mathcal{H}$ . If  $A$  is a quadratic form on  $\mathcal{H}$ , we denote by  $Q(A)$  its form domain and the value of  $A$  will be denoted by  $(\Psi|A\Phi)_{\mathcal{H}}$  for  $\Psi, \Phi \in Q(A)$ . We use the same notation for the quadratic form associated to a self-adjoint operator  $A$ , with domain  $Q(A) = \text{Dom}(|A|^{\frac{1}{2}})$ .

We now formulate the hypotheses of Thm. 2.1 below.

Let  $L^2(\mathbb{R}^d; d\mathbf{x})$  be the space of square integrable functions on  $\mathbb{R}^d$  with variable  $\mathbf{x} = (x_1, \dots, x_d)$ , and  $\mathcal{K}$  be a separable complex Hilbert space. We denote by  $\mathbf{p} = (p_1, \dots, p_d) = -i\nabla_{\mathbf{x}}$  the momentum operator on  $L^2(\mathbb{R}^d; d\mathbf{x})$ . The Hilbert space of the total system is:

$$\mathcal{H} := L^2(\mathbb{R}^d; d\mathbf{x}) \otimes \mathcal{K}$$

We fix a quadratic form  $H^0$  on  $\mathcal{H}$  and an external potential  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  which is a real Borel measurable function. The multiplication by  $V(\mathbf{x})$  is denoted by the same symbol.

The Hamiltonian of the system is obtained from the quadratic form on  $\mathcal{H}$  defined by

$$H^V := H^0 + V.$$

We assume the following conditions:

**(H.1)** There exists a dense domain  $\mathcal{D}_0$  such that

$$\mathcal{D}_0 \subseteq Q(H^0) \cap Q(V)$$

and  $H^V$  and  $H^0$  are closable and bounded from below on  $\mathcal{D}_0$ .

**(H.2)** There exist a vector of commuting self-adjoint operators  $\mathbf{P}_f = (P_{f,1}, \dots, P_{f,d})$  on  $\mathcal{K}$  such that  $H^0$  commutes with

$$\begin{aligned} \mathbf{P} &:= (P_1, \dots, P_d), \\ P_j &= \overline{p_j \otimes I + I \otimes P_{f,j}}, \end{aligned}$$

namely, for all  $\mathbf{k} \in \mathbb{R}^d$ ,  $e^{i\mathbf{k} \cdot \mathbf{P}} \mathcal{D}_0 = \mathcal{D}_0$  and it holds that

$$(e^{i\mathbf{k} \cdot \mathbf{P}} \Psi | H^0 e^{i\mathbf{k} \cdot \mathbf{P}} \Phi) = (\Psi | H^0 \Phi)$$

for all  $\Psi, \Phi \in \mathcal{D}_0$  and  $\mathbf{k} \in \mathbb{R}^d$ .

From (H.1),  $H^V$  and  $H^0$  are closable on  $\mathcal{D}_0$ , and we denote by  $\bar{H}^V$ ,  $\bar{H}^0$  the self-adjoint operators associated to the closure of  $H^V$ ,  $H^0$ . Let

$$E^V := \inf \sigma(\bar{H}^V) = \inf_{\Psi \in \mathcal{D}_0, \|\Psi\|=1} (\Psi | H^V \Psi)_{\mathcal{H}},$$

$$E^0 := \inf \sigma(\bar{H}^0) = \inf_{\Psi \in \mathcal{D}_0, \|\Psi\|=1} (\Psi | H^0 \Psi)_{\mathcal{H}}.$$

be the ground state energies. The key assumption of the main theorem is the following:

**(H.3)** There exist a measurable real function  $K(\mathbf{k})$  such that

$$\frac{1}{2} \{ \Omega(\mathbf{k}) + \Omega(-\mathbf{k}) - 2\Omega(0) \} \leq K(\mathbf{k}) \text{ on } \mathcal{D}_0, \forall \mathbf{k} \in \mathbb{R}^d,$$

where  $\Omega(\mathbf{k}) := e^{-i\mathbf{k} \cdot \mathbf{x}} H^0 e^{i\mathbf{k} \cdot \mathbf{x}}$ .

We set

$$h := K(\mathbf{p}) + V$$

which is a quadratic form on  $L^2(\mathbb{R}^d; d\mathbf{x})$ . We assume that

**(H.4)** There exists a non-trivial subspace  $\mathcal{D}_1$  of  $L^2(\mathbb{R}^d; d\mathbf{x})$  with  $\mathcal{D}_1 \subset Q(K(\mathbf{p})) \cap Q(V)$  such that for all  $f \in \mathcal{D}_1$  and  $\Psi \in \mathcal{D}_0$ ,  $f(\mathbf{x})\Psi \in \mathcal{D}_0$ . Moreover  $\mathcal{D}_1$  is invariant under the complex conjugation, i.e.  $\bar{f} \in \mathcal{D}_1$  for all  $f \in \mathcal{D}_1$ .

We define

$$e_0 := \inf_{f \in \mathcal{D}_1, \|f\|=1} (f | hf)_{L^2}.$$

The main theorem in this paper is the following.

**Theorem 2.1.** *Assume the hypotheses (H.1)–(H.4). Then the inequality*

$$E^V \leq E^0 + e_0$$

*holds. In particular, if  $e_0 < 0$ , then  $E_{\text{bin}} \geq -e_0 > 0$ .*

### 3. PROOF OF THEOREM 2.1

For arbitrary small  $\epsilon$ , we choose normalized vectors  $F \in \mathcal{D}_0$ ,  $f \in \mathcal{D}_1$  such that

$$(F | H^0 F)_{\mathcal{H}} \leq E^0 + \epsilon,$$

$$(f | hf)_{L^2} \leq e_0 + \epsilon.$$

Since by (H.4)  $h$  commutes with the complex conjugation, the function  $f$  can be chosen to be real. We consider the following extended Hilbert space

$$\mathcal{H}_{\text{ex}} := L^2(\mathbb{R}^d; d\mathbf{y}) \otimes \mathcal{H},$$

which naturally identified with the sets of  $\mathcal{H}$ -valued square integrable functions  $L^2((\mathbb{R}^d; d\mathbf{y}); \mathcal{H})$ . For  $\mathbf{y} \in \mathbb{R}^d$ , we set  $F_{\mathbf{y}} := e^{i\mathbf{y} \cdot \mathbf{p}} F$  and consider the  $\mathcal{H}$ -valued function:

$$\Phi : \mathbb{R}^d \ni \mathbf{y} \mapsto \Phi_{\mathbf{y}} := f(\mathbf{x}) F_{\mathbf{y}} \in \mathcal{H}.$$

The theorem will follow easily from the following three claims:

$$(5) \quad \Phi \in \mathcal{H}_{\text{ex}}, \|\Phi\| = 1,$$

$$(6) \quad \Phi \in Q(I \otimes H^0), (\Phi|I \otimes H^0\Phi)_{\mathcal{H}_{\text{ex}}} \leq (F|H^0F)_{\mathcal{H}} + (f|K(\mathbf{p})f)_{L^2},$$

$$(7) \quad \Phi \in Q(I \otimes V), (\Phi|I \otimes V\Phi)_{\mathcal{H}_{\text{ex}}} = (f|Vf)_{L^2}.$$

Let us first prove (5), (6) and (7). We have:

$$\begin{aligned} \int_{\mathbb{R}^d} \|\Phi_{\mathbf{y}}\|_{\mathcal{H}}^2 d\mathbf{y} &= \int_{\mathbb{R}^d} \|f(\mathbf{x})e^{i\mathbf{y} \cdot \mathbf{P}}F\|_{\mathcal{H}}^2 d\mathbf{y} = \int_{\mathbb{R}^d} \|e^{-i\mathbf{y} \cdot \mathbf{P}}f(\mathbf{x})e^{i\mathbf{y} \cdot \mathbf{P}}F\|_{\mathcal{H}}^2 d\mathbf{y} \\ &= \int_{\mathbb{R}^d} |f(\mathbf{x} - \mathbf{y})|^2 d\mathbf{y} \|F\|_{\mathcal{H}}^2 = \|f\|_{L^2(\mathbb{R}^d)}^2 \cdot \|F\|_{\mathcal{H}}^2 = 1, \end{aligned}$$

which proves (5). Since  $H^0$  is bounded below, (6) will follow from

$$(8) \quad (\Phi|I \otimes H^0\Phi)_{\mathcal{H}_{\text{ex}}} = \int_{\mathbb{R}^d} (\Phi_{\mathbf{y}}|H^0\Phi_{\mathbf{y}})_{\mathcal{H}} d\mathbf{y} \leq (F|H^0F)_{\mathcal{H}} + (f|K(\mathbf{p})f)_{L^2},$$

using that  $F \in Q(H^0)$  and  $f \in Q(K(\mathbf{p}))$ .

Denoting by  $\mathcal{F} : L^2(\mathbb{R}^d; d\mathbf{y}) \ni f \mapsto \hat{f} \in L^2(\mathbb{R}^d; d\mathbf{k})$  the unitary Fourier transform, we have:

$$\begin{aligned} \int_{\mathbb{R}^d} (\Phi_{\mathbf{y}}|H^0\Phi_{\mathbf{y}})_{\mathcal{H}} d\mathbf{y} &= \int_{\mathbb{R}^d} (f(\mathbf{x})F_{\mathbf{y}}|H^0f(\mathbf{x})F_{\mathbf{y}})_{\mathcal{H}} d\mathbf{y} \\ &= \int_{\mathbb{R}^d} (e^{-i\mathbf{y} \cdot \mathbf{P}}f(\mathbf{x})e^{i\mathbf{y} \cdot \mathbf{P}}F|e^{-i\mathbf{y} \cdot \mathbf{P}}H^0e^{i\mathbf{y} \cdot \mathbf{P}}e^{-i\mathbf{y} \cdot \mathbf{P}}f(\mathbf{x})e^{i\mathbf{y} \cdot \mathbf{P}}F)_{\mathcal{H}} d\mathbf{y} \\ &= \int_{\mathbb{R}^d} (f(\mathbf{x} - \mathbf{y})F|H^0f(\mathbf{x} - \mathbf{y})F)_{\mathcal{H}} d\mathbf{y} \\ &= \int_{\mathbb{R}^d} (e^{i\mathbf{k} \cdot \mathbf{x}}\hat{f}(\mathbf{k})F|H^0e^{i\mathbf{k} \cdot \mathbf{x}}\hat{f}(\mathbf{k})F)_{\mathcal{H}} d\mathbf{k} \\ &= \int_{\mathbb{R}^d} |\hat{f}(\mathbf{k})|^2 (F|\Omega(\mathbf{k})F)_{\mathcal{H}} d\mathbf{k}. \end{aligned}$$

Since  $f$  is real valued, we have:

$$\begin{aligned} &\int_{\mathbb{R}^d} |\hat{f}(\mathbf{k})|^2 (F|\Omega(\mathbf{k})F)_{\mathcal{H}} d\mathbf{k} \\ &= \frac{1}{2} \int_{\mathbb{R}^d} |\hat{f}(\mathbf{k})|^2 (F|(\Omega(\mathbf{k}) + \Omega(-\mathbf{k}) - 2\Omega(0))F)_{\mathcal{H}} d\mathbf{k} + \|f\|^2 (F|H^0F)_{\mathcal{H}} \\ &\leq \|F\|^2 \int_{\mathbb{R}^d} |\hat{f}(\mathbf{k})|^2 K(\mathbf{k}) d\mathbf{k} + \|f\|^2 (F|H^0F)_{\mathcal{H}} \\ &= (F|H^0F)_{\mathcal{H}} + (f|K(\mathbf{p})f)_{L^2}, \end{aligned}$$

which proves (6).

Similarly we have

$$\begin{aligned} (\Phi|I \otimes V\Phi)_{\mathcal{H}_{\text{ex}}} &= \int_{\mathbb{R}^d} (f(\mathbf{x})F_{\mathbf{y}}|V(\mathbf{x})f(\mathbf{x})F_{\mathbf{y}})_{\mathcal{H}} d\mathbf{y} \\ &= \int_{\mathbb{R}^d} (e^{-i\mathbf{y} \cdot \mathbf{P}}f(\mathbf{x})F_{\mathbf{y}}|e^{-i\mathbf{y} \cdot \mathbf{P}}V(\mathbf{x})f(\mathbf{x})F_{\mathbf{y}})_{\mathcal{H}} d\mathbf{y} \\ &= \int_{\mathbb{R}^d} (f(\mathbf{x} - \mathbf{y})F|V(\mathbf{x} - \mathbf{y})f(\mathbf{x} - \mathbf{y})F)_{\mathcal{H}} d\mathbf{y} \\ &= (f|Vf)_{L^2} \|F\|^2 = (f|Vf)_{L^2}, \end{aligned}$$

which proves (7). From (5), (6) and (7) we obtain

$$E^V \leq (\Phi|I \otimes H^V\Phi)_{\mathcal{H}_{\text{ex}}} \leq (F|H^0F)_{\mathcal{H}} + (f|(K(\mathbf{p}) + V)f)_{L^2} \leq E^0 + e_0 + 2\epsilon.$$

Since  $\epsilon$  is arbitrary we obtain the theorem.  $\square$

#### 4. EXAMPLES

In this section we give some examples to which Thm. 2.1 can be applied. If  $\mathfrak{h}$  is a Hilbert space, we denote by

$$\Gamma_s(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \otimes_s^n \mathfrak{h}$$

the bosonic Fock space over  $\mathfrak{h}$ . The vacuum vector in  $\Gamma_s(\mathfrak{h})$  will be denoted by  $\Omega$ ,  $a^*(h)$ ,  $a(h)$  for  $h \in \mathfrak{h}$  denote the creation/annihilation operators.

**4.1. Semi-relativistic Pauli-Fierz Hamiltonians.** The *semi-relativistic Pauli-Fierz Hamiltonian* is defined as follows: we take  $d = 3$  and

$$\begin{aligned} \mathcal{K} &= \Gamma_s(L^2(\mathbb{R}^3 \times \{1, 2\})), \\ H^V &= H_{\text{SRPF}}^V := \sqrt{(\mathbf{p} \otimes I + \sqrt{\alpha} \mathbf{A}(\mathbf{x}))^2 + m^2} - m + I \otimes H_f + V \otimes I, \end{aligned}$$

where  $\alpha \in \mathbb{R}$  is a coupling constant and  $m > 0$  is the mass of the electron (see [1]). The quantized vector potential  $\mathbf{A}(\mathbf{x})$  is defined by

$$(9) \quad \mathbf{A}(\mathbf{x}) = \frac{1}{\sqrt{2}} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} d\mathbf{k} \frac{\Lambda(\mathbf{k})}{|\mathbf{k}|^{1/2}} \mathbf{e}^{(\lambda)}(\mathbf{k}) (e^{i\mathbf{k} \cdot \mathbf{x}} a_{\lambda}(\mathbf{k}) + e^{-i\mathbf{k} \cdot \mathbf{x}} a_{\lambda}^*(\mathbf{k})),$$

where  $a_{\lambda}^*(\mathbf{k})$ ,  $a_{\lambda}(\mathbf{k})$  are creation and annihilation operators on  $\mathcal{K}$ ,  $\Lambda$  is a real-function such that  $\Lambda, |\mathbf{k}|^{-1/2} \Lambda \in L^2(\mathbb{R}^3)$  and the polarization vectors  $\mathbf{e}^{(\lambda)} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  satisfy

$$\mathbf{e}^{(\lambda)}(\mathbf{k}) \cdot \mathbf{e}^{(\lambda')}(\mathbf{k}) = \delta_{\lambda, \lambda'}, \quad \mathbf{e}^{(\lambda)}(\mathbf{k}) \cdot \mathbf{k} = 0.$$

The free photon energy  $H_f$  is defined by

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |\mathbf{k}| a_{\lambda}^*(\mathbf{k}) a_{\lambda}(\mathbf{k}) d\mathbf{k}$$

Let

$$\mathcal{F}_{\text{fin}} := \mathcal{L}[\{a^*(f_1) \cdots a^*(f_n) \Omega_{\text{photon}}, \Omega_{\text{photon}} | f_j \in C_0^\infty(\mathbb{R}^3 \times \{1, 2\}), j = 1, 2, \dots, n, n \in \mathbb{N}\}]$$

be a finite photon subspace where  $\Omega_{\text{photon}} = (1, 0, 0, \dots) \in \mathcal{K}$ . We set

$$(10) \quad \mathcal{D}_0 = C_0^\infty(\mathbb{R}^3) \hat{\otimes} \mathcal{F}_{\text{fin}},$$

$$(11) \quad \mathbf{P}_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \mathbf{k} a_{\lambda}^*(\mathbf{k}) a_{\lambda}(\mathbf{k}) d\mathbf{k}$$

where  $\hat{\otimes}$  indicates the algebraic tensor product. Then the above operator satisfy the condition (H.1) and (H.2). Moreover, it is proved that (H.3) holds with  $K(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2} - m$  (see

[1]). We assume that  $V \in L^1_{\text{loc}}(\mathbb{R}^3; d\mathbf{x})$  and set  $\mathcal{D}_1 = C_0^\infty(\mathbb{R}^3)$ . Then (H.4) holds. Therefore  $E_{\text{SRPF}}^V \leq E_{\text{SRPF}}^0 + e_0$  holds with

$$E_{\text{SRPF}}^\# := \inf_{\Psi \in \mathcal{D}_0, \|\Psi\|=1} (\Psi | H_{\text{SRPF}}^V \Psi)_{\mathcal{H}}, \quad \# = V, 0$$

$$e_0 = \inf_{f \in C_0^\infty, \|f\|=1} (f | (\sqrt{\mathbf{p}^2 + m^2} - m + V)f)_{L^2}.$$

**4.2. Pauli-Fierz Hamiltonian with dipole approximation.** The Pauli-Fierz Hamiltonian with dipole approximation is defined by

$$(12) \quad H_{\text{DP}}^V = \frac{1}{2m}(\mathbf{p} \otimes I + \sqrt{\alpha} \mathbf{A}(0))^2 + I \otimes H_f + V \otimes I,$$

where  $\mathbf{A}(0)$  is defined in (9) with  $\mathbf{x} = 0$ .  $H_{\text{DP}}^V$  is defined on  $\mathcal{D}_0 = C_0^\infty \hat{\otimes} \mathcal{F}_{\text{fin}}$ . Clearly (H.1) holds. The operator  $H_{\text{DP}}^0$  is not translation invariant, but it preserves the particle momentum  $\mathbf{p}$ . Hence we set

$$\mathbf{P}_f = 0, \quad \mathbf{P} = \mathbf{p}.$$

Then (H.2) holds. For this Hamiltonian, we have

$$\frac{1}{2}(\Omega(\mathbf{k}) + \Omega(-\mathbf{k}) - 2\Omega(0)) = \frac{\mathbf{k}^2}{2m},$$

which implies that (H.3) holds with  $K(\mathbf{k}) = \mathbf{k}^2/2m$ . We assume that  $V \in L^1_{\text{loc}}(\mathbb{R}^3)$  and set  $\mathcal{D}_1 = C_0^\infty(\mathbb{R}^3)$ . Then (H.4) holds. Therefore the inequality  $E_{\text{DP}}^V \leq E_{\text{DP}}^0 + e_0$  holds with

$$E_{\text{DP}}^\# := \inf_{\Psi \in \mathcal{D}_0, \|\Psi\|=1} (\Psi | H_{\text{DP}}^V \Psi)_{\mathcal{H}}, \quad \# = V, 0$$

$$e_0 = \inf_{f \in C_0^\infty, \|f\|=1} (f | (\frac{\mathbf{p}^2}{2m} + V)f)_{L^2}.$$

**4.3. Nelson type Hamiltonians.** We define the Nelson type Hamiltonian by

$$\mathcal{K} = \Gamma_s(L^2(\mathbb{R}^d)),$$

$$H^V = H_{\text{Nel}}^V := B(\mathbf{p}^2) \otimes I + I \otimes H_f + P(\phi(\mathbf{x})),$$

where  $B : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a Bernstein function, i.e.,

$$(13) \quad B(u) \geq 0, \quad B(0) = 0, \quad (-1)^n \frac{d^n B(u)}{du^n} \geq 0, \quad n = 1, 2, \dots$$

The field operator  $\phi(\mathbf{x})$  is defined by

$$\phi(\mathbf{x}) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} (g(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} a^*(\mathbf{k}) + \overline{g(\mathbf{k})} e^{i\mathbf{k} \cdot \mathbf{x}} a(\mathbf{k})) d\mathbf{k}$$

with  $g \in L^2(\mathbb{R}^d)$ ,  $a^*, a$  are creation and annihilation operators on  $\mathcal{K}$  and  $P$  is a real, bounded below polynomial.

The free boson Hamiltonian  $H_f$  is defined by

$$H_f = \int_{\mathbb{R}^d} \omega(\mathbf{k}) a^*(\mathbf{k}) a(\mathbf{k}) d\mathbf{k},$$

where  $\omega$  is a non-negative function. We refer the reader to [2] for a recent study of the Nelson-type Hamiltonians with Bernstein function type kinetic energy. We set

$$\begin{aligned}\mathcal{F}_{\text{fin}} &:= \mathcal{L}[\{a^*(f_1) \cdots a^*(f_n)\Omega_{\text{b}}, \Omega_{\text{b}}|f_j \in C_0^\infty(\mathbb{R}^3), j = 1, 2, \dots, n, n \in \mathbb{N}\}], \\ \mathcal{D}_0 &= C_0^\infty(\mathbb{R}^d) \hat{\otimes} \mathcal{F}_{\text{fin}}\end{aligned}$$

where  $\Omega_{\text{b}} = (1, 0, 0, \dots) \in \mathcal{K}$ . Assume that  $V \in L_{\text{loc}}^1(\mathbb{R}^d)$ . By (13), we have

$$B(u) \leq \frac{u^3}{6} + B''(0)\frac{u^2}{2} + B'(0)u.$$

Hence,  $C_0^\infty(\mathbb{R}^d) \subset \text{Dom}(B(\mathbf{p}^2))$ . Then  $H_{\text{Nel}}^V$  and  $H_{\text{Nel}}^0$  are well-defined on  $\mathcal{D}_0$  and (H.1) holds. We set

$$\mathbf{P}_f = \int_{\mathbb{R}^d} \mathbf{k} a^*(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}.$$

Then,  $H_{\text{Nel}}^0$  commutes with  $P_j = \overline{p_j \otimes I + I \otimes P_{f,j}}, j = 1, \dots, d$  and (H.2) holds. Next we check (H.3). We note that

$$\Omega(\mathbf{k}) + \Omega(-\mathbf{k}) - 2\Omega(0) = B((\mathbf{p} + \mathbf{k})^2) + B((\mathbf{p} - \mathbf{k})^2) - 2B(\mathbf{p}^2).$$

We have the following lemma:

**Lemma 4.1.** *For all  $\mathbf{p}, \mathbf{k} \in \mathbb{R}^d$ , the inequality*

$$\frac{1}{2} (B((\mathbf{p} + \mathbf{k})^2) + B((\mathbf{p} - \mathbf{k})^2) - 2B(\mathbf{p}^2)) \leq B(\mathbf{k}^2).$$

*holds.*

*Proof.* It is known that any Bernstein function can be written in the form

$$(14) \quad B(u) = a + bu + \int_{\mathbb{R}_+} (1 - e^{-tu}) \mu(dt), \quad (u \geq 0)$$

where  $a, b$  are non-negative constants and  $\mu$  is a non-negative measure on  $\mathbb{R}_+$  such that  $\int_{\mathbb{R}_+} \min\{t, 1\} \mu(dt) < \infty$  (see [2]). Hence it is sufficient to prove the inequality

$$(15) \quad -e^{-(\mathbf{p}+\mathbf{k})^2 t} - e^{-(\mathbf{p}-\mathbf{k})^2 t} + 2e^{-\mathbf{p}^2 t} \leq 2(1 - e^{-\mathbf{k}^2 t}),$$

for all  $t \geq 0$  and  $p, k \in \mathbb{R}^d$ . If  $t = 0$ , (15) is trivial. Without loss of generality, one can set  $t = 1$ . Moreover we can assume that  $k = (\kappa, 0, 0)$ ,  $\kappa \geq 0$  by the spherical symmetry of (15). Then (15) will follow from

$$b_\kappa(p_1) := -e^{-(p_1+\kappa)^2} - e^{-(p_1-\kappa)^2} + 2e^{-p_1^2} \leq 2(1 - e^{-\kappa^2}),$$

where  $\mathbf{p} = (p_1, p_2, p_3)$ . It is enough to show that  $b_\kappa(p_1) \leq 2(1 - e^{-\kappa^2})$  for  $\kappa > 0$  and  $p_1 > 0$ . We set  $p_1 = a\kappa$  with  $a > 0$ . Then

$$\begin{aligned} b_\kappa(p_1) &= e^{-a^2\kappa^2} \left[ -e^{-\kappa^2}(e^{-2a\kappa^2} + e^{2a\kappa^2}) + 2 \right] \\ &\leq e^{-a^2\kappa^2} \left[ -2e^{-\kappa^2} + 2 \right] \\ &\leq 2(1 - e^{-\kappa^2}), \end{aligned}$$

where we used the inequality  $e^{-2a\kappa^2} + e^{2a\kappa^2} \geq 2$  and  $e^{-a^2\kappa^2} \leq 1$ .  $\square$

Lemma 4.1 implies that (H.3) holds with  $K(\mathbf{k}) = B(\mathbf{k}^2)$ . By setting  $\mathcal{D}_1 = C_0^\infty(\mathbb{R}^d)$ , (H.4) holds. Therefore, by Theorem 2.1,  $E_{\text{Nel}}^V \leq E_{\text{Nel}}^0 + e_0$  holds with

$$\begin{aligned} E_{\text{Nel}}^\sharp &:= \inf_{\Psi \in \mathcal{D}_0, \|\Psi\|=1} (\Psi | H_{\text{Nel}}^V \Psi)_{\mathcal{H}}, \quad \sharp = V, 0 \\ e_0 &:= \inf_{f \in \mathcal{D}_1, \|f\|=1} (f | (B(\mathbf{p}^2) + V)f)_{L^2}. \end{aligned}$$

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